問題

(1) 変数変換を使って 1 変数ガウス分布 (1.46) が (1.49) を満たすことを確かめよ。

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{2\pi\sigma^2}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\} \tag{1.46}$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty}\mathcal{N}(x|\mu,\sigma^2)xdx = \mu \tag{1.49}$$

(2) 次に、規格化条件

$$\int_{-\infty}^{\infty}\mathcal{N}(x|\mu,\sigma^2)dx = 1 \tag{1.127}$$

の両辺を σ^2 に関して微分し、ガウス分布が (1.50) を満たすことを確かめよ。

$$\mathbb{E}[x^2] = \int_{-\infty}^{\infty}\mathcal{N}(x|\mu,\sigma^2)x^2dx = \mu^2+\sigma^2 \tag{1.50}$$

(3) 最後に (1.51) が成り立つことを示せ。

$$var[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2 \tag{1.51}$$

解 (1)

$$\begin{aligned}\int_{-\infty}^{\infty}\mathcal{N}(x|\mu,\sigma^2)xdx &= \int_{-\infty}^{\infty}\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}xdx \\ &= \frac{1}{(2\pi\sigma^2)^{1/2}}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}xdx\end{aligned}$$

ここで $\dfrac{x-\mu}{\sigma} = t$ とすると

$$x-\mu = \sigma t$$

$$x = \mu+\sigma t$$

$$dx = \sigma dt$$

x	$-\infty$	$\to$	∞
t	$-\infty$	$\to$	∞

となるので

$$\begin{aligned}&= \frac{1}{(2\pi\sigma^2)^{1/2}}\int_{-\infty}^{\infty}\exp(-\frac{t^2}{2})(\mu+\sigma t)\sigma dt \\ &= \frac{\sigma}{(2\pi)^{1/2}\sigma}\int_{-\infty}^{\infty}\exp(-\frac{t^2}{2})(\mu+\sigma t)dt \\ &= \frac{1}{(2\pi)^{1/2}}\Big\{\mu\int_{-\infty}^{\infty}\exp(-\frac{t^2}{2})dt + \sigma\int_{-\infty}^{\infty}\exp(-\frac{t^2}{2})tdt\Big\}\end{aligned}$$

括弧内の第一項と第二項についてそれぞれ計算すると

$$\begin{aligned}第一項 &= \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2})dt \\ &= (2\pi)^{1/2}\end{aligned}$$

$$第二項 = \int_{-\infty}^{\infty} t\exp(-\frac{t^2}{2})dt$$

ここで

$$f(t) = t\exp(-\frac{t^2}{2})$$

とすると

$$\begin{aligned}f(-t) &= -t\exp\big(-\frac{(-t)^2}{2}\big) \\ &= -t\exp\big(-\frac{t^2}{2}\big) \\ &= -f(t)\end{aligned}$$

となり、$f(t)$ は奇関数だとわかる。

よって

$$\begin{aligned}第二項 &= \int_{-\infty}^{\infty} t\exp(-\frac{t^2}{2})dt \\ &= \int_{-\infty}^{0} t\exp(-\frac{t^2}{2})dt + \int_{0}^{\infty} t\exp(-\frac{t^2}{2})dt \\ &= -\int_{0}^{\infty} t\exp(-\frac{t^2}{2})dt + \int_{0}^{\infty} t\exp(-\frac{t^2}{2})dt \\ &= 0\end{aligned}$$

これらの値を代入すると

$$\frac{1}{(2\pi)^{1/2}}\Big\{\mu\int_{-\infty}^{\infty} \exp(-\frac{t^2}{2})dt + \sigma\int_{-\infty}^{\infty} \exp(-\frac{t^2}{2})tdt\Big\}$$

$$\begin{aligned}&= \frac{1}{(2\pi)^{1/2}} \times \mu \times (2\pi)^{1/2} \\ &= \mu\end{aligned}$$

$\therefore$ 式 (1.46) が式 (1.49) を満たすことが示された。

解 (2)

正規分布の規格化条件

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)dx = 1 \tag{1.127}$$

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}$$

代入すると

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}dx \\
&= \frac{1}{(2\pi\sigma^2)^{1/2}}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}dx \\
&= 1
\end{aligned}$$

移項すると

$$\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}dx = (2\pi\sigma^2)^{1/2} \tag{1}$$

両辺を σ^2 で微分する

$$\begin{aligned}
\text{左辺} &= \Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}'\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}dx \\
&= (-\frac{1}{2})(\frac{1}{\sigma^2})'\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx \\
&= (-\frac{1}{2})(-\sigma^2)^{-2}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx \\
&= \frac{1}{2\sigma^4}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx \\
\text{右辺} &= (2\pi\sigma^2)'\times(2\pi\sigma^2)^{1/2\,\prime} \\
&= 2\pi\times\frac{1}{2}(2\pi\sigma^2)^{-(1/2)} \\
&= \pi\times\frac{1}{(2\pi\sigma^2)^{1/2}} \\
&= \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}}
\end{aligned}$$

(1) に戻すと

$$\frac{1}{2\sigma^4}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx = \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}}$$

両辺に $\dfrac{\sigma^2\sqrt{2\sigma^2}}{\sqrt{\pi}}$ をかけると

$$\frac{\sqrt{2\sigma^2}}{2\sigma^2\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx = \sigma^2$$

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx = \sigma^2$$

ここで、式 (1.50) と見比べると

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}(x-\mu)^2dx = \mathbb{E}[(x-\mu)^2]$$

$$\therefore \mathbb{E}[(x-\mu)^2] = \sigma^2$$

次に $\mathbb{E}[(x-\mu)^2] = \sigma^2$ の左辺を展開する

$$\begin{aligned}\text{左辺} &= \mathbb{E}[x^2 - 2\mu x + \mu^2] \\ &= \mathbb{E}[x^2] - 2\mu\mathbb{E}[x] + \mu^2 \\ &= \mathbb{E}[x^2] - 2\mu^2 + \mu^2 \\ &= \mathbb{E}[x^2] - \mu^2\end{aligned}$$

$$\mathbb{E}[x^2] - \mu^2 = \sigma^2$$

$$\therefore \mathbb{E}[x^2] = \mu^2 + \sigma^2$$

解 (3)

$$\begin{aligned}var[x] &= \mathbb{E}[x^2] - \mathbb{E}[x]^2 \\ &= (\sigma^2 + \mu^2) - (\mu)^2 \\ &= \sigma^2\end{aligned}$$